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THE AVERAGING METHOD FOR ASYMPTOTIC EVOLUTIONS I : STOCHASTIC DIFFERENTIAL EQUATIONS

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ABSTRACT: Asymptotic evolutions of open systems are studied. Conditions are given under which successive approximate evolutions obtained by the method of averaging are asymptotic to the exact evolution of the open system. It is shown that these conditions are satisfied in the case of stochastic differential equations which describe the evolution of spins in random magnetic fields.

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§0. INTRODUCTION

Asymptotic evolutions of open systems have been studied rigorously in the weak coupling limit [1, 2, 3]. The averaging method has been used to calculate successive terms in a formal series for the asymptotic evolution [4], but the question in which sense higher order terms improve the weak coupling result has remained unanswered. In this paper we attempt to give an answer. In section 1 we motivate the work by describing informally two examples. In Section 2 we state the main results, which make precise the sense in which successive approximate evolutions obtained by the averaging method are asymptotic to the exact evolution of the open system, and show how the weak coupling results may be recovered. These results are proved in Section 3, under conditions which are stated abstractly. In Section 4 we show that these conditions hold in the case of a multiplicative stochastic differential equation which was treated formally in [4]. Applications to quantum open systems will be given in [5].

§1. MOTIVATION

The following mathematical problem occurs in various guises in statistical physics. We have an evolution equation

$$\frac{d}{dt} f^{\lambda}(t) = \lambda A(t) f^{\lambda}(t) \tag{1.1}$$

on a vector space $\mathcal{B}$, and start with initial data f_0 in a subspace $\mathcal{B}_0$; we project the solution $f^{\lambda}(t)$ back into $\mathcal{B}_0$ with a projection operator P_0 and ask for the behaviour of $P_0 f^{\lambda}(t)$ for large t. We have in mind a situation in which the space $\mathcal{B}$ is associated with a physical system coupled to a reservoir, the subspace $\mathcal{B}_0$ is associated with the system alone and projection on $\mathcal{B}_0$ corresponds to an averaging over the reservoir. There is a relaxation time $1/\alpha$ associated with the reservoir, in addition to the characteristic time $1/\lambda$ associated with the coupled system. It sometimes happens that when the ratio λ/α is small, we have physical grounds for suspecting that there is a time-independent operator G^{λ} on $\mathcal{B}_0$ such that

$$P_0 f^{\lambda}(t) \sim e^{G^{\lambda} t} f_0 .$$

The mathematical problem then has three parts: find G^{λ}, make precise the sense in which the approximation holds, and investigate higher order corrections to exponential behaviour.

Consider the following example: A spin-$\frac{1}{2}$ particle, with magnetic moment

$$\vec{\mu} = \frac{\hbar}{2} \gamma \vec{\sigma},$$

is placed in a random magnetic field $\vec{\mathcal{H}}(t)$. The density matrix ρ describing the state of the system is assumed to develop in time according to the Schrodinger equation of motion

$$\frac{d}{dt} \rho(t) = \frac{i}{\hbar} [\rho(t), H(t)] , \tag{1.3}$$

with

$$H(t) = -\vec{\mu} \cdot \vec{\mathcal{H}}(t) . \tag{1.4}$$

The density matrix $\rho(t)$ may be parametrized by the polarization vector $\vec{p}(t)$ as

$$\rho(t) = \tfrac{1}{2}(1 + \vec{p}(t) \cdot \vec{\sigma}) , \quad \vec{p}(t) = \mathrm{trace}[\rho(t) \vec{\sigma}] . \tag{1.5}$$

Using (1.2) - (1.5) we get the evolution equation

$$\frac{d}{dt} \vec{p}(t) = -\gamma \vec{\mathcal{H}}(t) \wedge \vec{p}(t) . \tag{1.6}$$

We shall consider two cases: In the first, $\vec{\mathcal{H}}(t)$ is in a fixed direction $\vec{k}$, so that $\vec{\mathcal{H}}(t) = \mathcal{H}(t)\vec{k}$ and $\mathcal{H}(t)$ is a zero-mean stationary Gaussian random process with covariance

$$k(t) = \langle \mathcal{H}(s+t) \mathcal{H}(s) \rangle . \tag{1.7}$$

Then $\vec{p}(t) \cdot \vec{k}$ is constant and the projection $p^{\perp}(t)$ of $\vec{p}(t)$ on the plane perpendicular to $\vec{k}$ is a stochastic process satisfying

$$\frac{d}{dt} p^{\perp}(t) = \gamma \mathcal{H}(t) J p^{\perp}(t) , \tag{1.8}$$

where J is a skew-adjoint matrix such that $J^2 = -1$.

The solution of (1.8) satisfying $p^{\perp}(0) = p_0^{\perp}$ is given by

$$p^{\perp}(t) = \exp\left[\gamma \int_0^t \mathcal{H}(s)\,ds\, J\right] p_0^{\perp}. \tag{1.9}$$

In this simple example we can compute $\langle p^{\perp}(t)\rangle$ by expanding the exponential, averaging each term (using (1.7) and the Gaussian property) and re-summing. We get

$$\langle p^{\perp}(t)\rangle = \exp\left[-\gamma^2 \int_0^t S(u)\,du\right] p_0^{\perp}, \tag{1.10}$$

where

$$S(u) = \int_0^u k(s)\,ds. \tag{1.11}$$

To investigate the large time behaviour of $\langle p^{\perp}(t)\rangle$ we introduce the time average $\bar{S}$ of $S(t)$,

$$\bar{S} = \lim_{T\to\infty} \frac{1}{T}\int_0^T S(t)\,dt, \tag{1.12}$$

and put

$$M(t) = \gamma^2 \int_0^t [\bar{S} - S(u)]\,du \tag{1.13}$$

so that

$$\langle p^{\perp}(t)\rangle = \exp[-\gamma^2 \bar{S} t + M(t)]\, p_0^{\perp}. \tag{1.14}$$

The special case $k(t) = e^{-\alpha|t|}$ is instructive: we have

$$\bar{S} = 1/\alpha, \quad M(t) = (\gamma/\alpha)^2 (1 - e^{-\alpha t}).$$

When γ/α is small, we expand $\exp M(t)$ as a power series retaining only the first two terms:

$$\langle p^{\perp}(t)\rangle \sim (1 + M(t))\, e^{-\gamma^2 \bar{S} t}\, p_0^{\perp}. \tag{1.15}$$

Notice that, with these assumptions, $\langle p^{\perp}(t)\rangle$ decays on the time scale α/γ^2, while $M(t)$ becomes a constant on the time scale $1/\alpha$; thus we have separated the long-time behaviour of $\langle p^{\perp}(t)\rangle$ from the short-time corrections to exponential decay. We shall return to this point at the end of §2.

The second case we consider is when all three components of $\vec{\mathcal{H}}(t)$ are independent, identically - distributed stochastic processes. In this case we have

$$\frac{d}{dt}\vec{p}(t) = \gamma(\vec{\mathcal{H}}(t)\cdot\vec{J})\,\vec{p}(t), \tag{1.16}$$

where J_1, J_2, J_3 are skew-symmetric generators of the rotation group. The solution of (1.16) satisfying $\vec{p}(0) = \vec{p}_0$ can be written as

$$\begin{aligned}\vec{p}(t) &= T\exp\left[\gamma \int_0^t \vec{\mathcal{H}}(s)\cdot\vec{J}\,ds\right]\vec{p}_0 \\ &= \sum_{n=0}^{\infty} \gamma^n \int\cdots\int_{t\geq t_1\geq\cdots\geq t_n\geq 0} (\vec{\mathcal{H}}(t_1)\cdot\vec{J})\cdots(\vec{\mathcal{H}}(t_n)\cdot\vec{J})\,\vec{p}_0\, dt_n\cdots dt_1 .\end{aligned} \tag{1.17}$$

If we assume that the components of $\vec{\mathcal{H}}(t)$ are mean-zero *stationary*

Gaussian processes with

$$\langle \mathcal{H}_i(s+t)\mathcal{H}_j(s)\rangle = \delta_{ij} k(t), \tag{1.18}$$

we can attempt to compute $\langle \vec{p}(t)\rangle$ by averaging (1.17) term by term; at eighth order in γ the prospect of discerning a pattern looks hopeless [6].

An alternative approach to the investigation of $\langle \vec{p}(t)\rangle$ is the method of averaging, which was used in [4] in a problem, a special case of which is mathematically identical to the one considered here.

In classical mechanics, the method of averaging has a long history (see [7]); the standard account is the book of Bogoliubov and Mitropolsky [8]. Its use in quantum mechanical problems was advocated by Case [9] and in stochastic problems by Ford [10] (see also [4]). With the benefit of hindsight, we give a simplified version of the scheme as presented in [4].

Consider the equation (1.1). The first example we discussed suggests that it may be possible to find operators $M^\lambda(t)$ and G^λ on $\mathcal{B}_0$ such that

$$P_0 f^\lambda(t) \sim (1+M^\lambda(t))\, e^{G^\lambda t} f_0 ,$$

when λ is small compared with some characteristic inverse time α (cf eq. (1.15)). To find $M^\lambda(t)$ and G^λ we first put

$$f^\lambda(t) = (1+F^\lambda(t))\, e^{G^\lambda t} f_0 , \tag{1.19}$$

where $F^\lambda(t)$ is an operator from $\mathcal{B}_0$ into $\mathcal{B}$ and G^λ is a time-independent operator on $\mathcal{B}_0$. From (1.1) and (1.19) we have

$$G^\lambda + \dot{F}^\lambda(t) = \lambda A(t) P_0 + \lambda A(t) F^\lambda(t) - F^\lambda(t) G^\lambda . \tag{1.20}$$

We attempt to solve this using formal power series

$$F^\lambda(t) = \lambda F^{(1)}(t) + \lambda^2 F^{(2)}(t) + \cdots , \tag{1.21}$$

$$G^\lambda = \lambda G^{(1)} + \lambda^2 G^{(2)} + \cdots , \tag{1.22}$$

and equating terms of the same order in λ in (1.20). Thus we get the hierarchy of equations

$$\left.\begin{aligned} G^{(1)} + \dot{F}^{(1)}(t) &= A(t) P_0 , \\ G^{(r)} + \dot{F}^{(r)}(t) &= A(t) F^{(r-1)}(t) - \sum_{s=1}^{r-1} F^{(r-s)}(t)\, G^{(s)} . \end{aligned}\right\} \tag{1.23}$$

Since $f^\lambda(0) = f_0$, we require $F^{(r)}(0) = 0$ for all r, but there is still great arbitrariness in the choice of $F^\lambda(t)$ and G^λ. The core of the method of averaging is the introduction of the averaging operation $\mathcal{E}$ on time-dependent operators given by

$$\mathcal{E}(B(\cdot)) = \lim_{T\to\infty} \frac{1}{T}\int_0^T P_0 B(t) P_0 \, dt \tag{1.24}$$

and the imposition of the requirement that

$$\mathcal{E}(G^{(r)}) = G^{(r)} \ , \ \mathcal{E}(\dot{F}^{(r)}) = 0 \ , \ r = 1,2,\dots \ . \tag{1.25}$$

We shall see that under these assumptions the hierarchy has a unique solution, which can be obtained recursively. Denote by $F^{\lambda}_{n-1}(t)$ and G^{λ}_{n} the polynomials of order n-1 and n in λ got by truncating the formal power series (1.21) and (1.22). In favourable cases it is possible to obtain norm-estimates of the difference $P_0 f^{\lambda}(t) - (1+P_0F^{\lambda}_{n-1}(t))\, e^{G^{\lambda}_n t}$ which enable us to regard $(1+P_0F^{\lambda}_{n-1}(t))\, e^{G^{\lambda}_n t}$ as an asymptotic approximation of order n to $P_0 f^{\lambda}(t)$ for λ/α small.

The method can be applied to our second example: Let $\{\mathcal{H}_i(t) : i = 1,2,3; t \in \mathbb{R}\}$ be independent stationary Gaussian processes on a probability space (Ω,μ) and assume that the covariance matrix is $\langle \mathcal{H}_i(s+t)\mathcal{H}_j(s)\rangle = \delta_{ij}\, e^{-\alpha|t|}$. Equation (1.16) is a differential equation of the form (1.1) on the space $\mathcal{B} = L^2(\Omega,\mathbb{R}^3,\mu)$ of square-integrable functions from (Ω,μ) to $\mathbb{R}^3$; the subspace $\mathcal{B}_0$ consists of the constant functions (which are vectors in $\mathbb{R}^3$), and the projection P_0 is the averaging over the stochastic process: $P_0 f = \int_\Omega f(\omega)\,\mu(d\omega) = \langle f\rangle$. The relaxation time of the reservoir is $1/\alpha$ and the characteristic time of the coupled system is $1/\gamma$. Application of the method up to fourth-order yields (see [4] and § 4 below)

$$G^{(1)} = P_0F^{(1)}(t) = 0 = P_0F^{(3)}(t) = G^{(3)}; \ \lambda^2 G^{(2)} = -2\gamma(\gamma/\alpha), \ \lambda^4 G^{(4)} = -2\gamma(\gamma/\alpha)^3;$$

$$\lambda^2 M^{(2)}(t) = \lambda^2 P_0 F^{(2)}(t) = (\gamma/\alpha)^2(1-e^{-\alpha t});$$

$$\|\langle\vec{p}(t)\rangle - \exp[-2\gamma(\gamma/\alpha)t]\,\vec{p}_0\| \leqslant (\gamma/\alpha)^2(a_2 + b_2\gamma^2 t/\alpha)\|\vec{p}_0\| ,$$

$$\|\langle\vec{p}(t)\rangle - \exp[-2\gamma\{(\gamma/\alpha)+(\gamma/\alpha)^3\}t]\,\vec{p}_0\| \leqslant (\gamma/\alpha)^2(c + b_4\gamma(\gamma/\alpha)^3 t)\|\vec{p}_0\|,$$

$$\|\langle\vec{p}(t)\rangle - [1+(\gamma/\alpha)^2(1-e^{-\alpha t})]\exp[-2\gamma\{(\gamma/\alpha)+(\gamma/\alpha)^3\}t]\,\vec{p}_0\| \leqslant (\gamma/\alpha)^4(a_4+b_4\gamma^2 t/\alpha)\|\vec{p}_0\|;$$

for some constants a_2, b_2, a_4, b_4, c.

2. STATEMENT OF MAIN RESULT AND DISCUSSIONS.

Let $\mathcal{B}$ be a Banach space, $\mathcal{B}_0$ a closed subspace of $\mathcal{B}$, P_0 a norm one projection of $\mathcal{B}$ onto $\mathcal{B}_0$, $P_1 = 1 - P_0$. Let $t \mapsto A(t)$ be a strongly continuous function on $\mathbb{R}^+$ with values in $\mathcal{L}(\mathcal{B})$, and consider the differential equation in $\mathcal{B}$:

$$\frac{d}{dt} f^{\lambda}(t) = \lambda A(t) f^{\lambda}(t) \ , \ t \geqslant 0. \tag{2.1}$$

Given the initial data f_0 in $\mathcal{B}$, the equation (2.1) has a unique continuous solution on any compact interval [0,T], hence a unique continuous solution on $[0,\infty)$, given by

$$f^{\lambda}(t) = U^{\lambda}(t,0) f_0 \ , \ t \geqslant 0 \tag{2.2}$$

where

$$U^{\lambda}(t,s) = T\exp[\lambda \int_s^t A(u)\,du]$$

$$= \sum_{n=0}^{\infty} \lambda^n \int_{t \geqslant u_1 \geqslant \cdots \geqslant u_n \geqslant s} \cdots \int A(u_1)\cdots A(u_n)\, du_n \cdots du_1 \ . \tag{2.3}$$

Assume

(*) $\|U^{\lambda}(t,s)\| = 1$ for all $t \geqslant s \geqslant 0$,

(**) $P_0 A(t_1)\cdots A(t_{2m+1}) P_0 = 0$ for all $m = 0,1,\dots$ and $t_1,\dots t_{2m+1} \geqslant 0$.

We look for an approximate expression of $P_0 f^\lambda(t)$ assuming $f^\lambda(0) = f_0$ to be in $\mathcal{B}_0$. Define

$$A_{ij}(t) = P_i A(t) P_j\,, \quad i,j = 0,1\,, \quad t \geq 0\,, \tag{2.4}$$

and assume that the expressions

$$K^{(2)}(t_1,t_2) = -A_{01}(t_1)A_{10}(t_2)\,, \quad t_1 \geq t_2\,, \tag{2.5}$$

$$K^{(4)}(t_1,t_4) = \iint_{t_1 \geq t_2 \geq t_3 \geq t_4} A_{01}(t_1)A_{11}(t_2)A_{11}(t_3)A_{10}(t_4)\, dt_3\, dt_2 \tag{2.6}$$

are functions of the time differences $t_1 - t_2,\, t_1 - t_4$ alone, then set

$$K^{(n)}(t-s) = K^{(n)}(t,s)\,, \quad n = 2,4\,, \quad t \geq s \geq 0\,. \tag{2.7}$$

The hierarchy of equations (1.23), subject to condition (1.25) with the averaging (1.24) becomes, up to fourth order,

$$\left.\begin{aligned} &\dot F^{(1)}(t) = A_{10}(t)\,, \quad G^{(2)} + \dot F^{(2)}(t) = A(t)F^{(1)}(t)\,, \\ &\dot F^{(3)}(t) = A(t)F^{(2)}(t) - F^{(1)}(t)G^{(2)},\; G^{(4)} + \dot F^{(4)}(t) = A(t)F^{(3)}(t) - F^{(2)}(t)G^{(2)}, \end{aligned}\right\} \tag{2.8}$$

where we have used (**) to find $A(t)P_0 = A_{10}(t)$ and $G^{(1)} = G^{(3)} = 0$. Solving the hierarchy (2.8) subject to condition (1.25) yields

$$G^{(2)} = -\int_0^\infty K^{(2)}(t)\,dt\,, \tag{2.9}$$

$$G^{(4)} = \int_0^\infty K^{(4)}(t)\,dt - \int_0^\infty K^{(2)}(t)\,dt \int_0^\infty K^{(2)}(s)\,ds\,, \tag{2.10}$$

$$M^{(2)}(t) = P_0 F^{(2)}(t) = \int_0^\infty K^{(2)}(s)\,s\,ds + \int_t^\infty (t-s)K^{(2)}(s)\,ds\,, \tag{2.11}$$

assuming the integrals to exist (see §3 for the details). Let f_0 be in $\mathcal{B}_0$, and let

$$x_2^\lambda(t) = y_2^\lambda(t) = \exp[\lambda^2 G^{(2)} t]\, f_0\,, \tag{2.12}$$

$$x_4^\lambda(t) = \exp[(\lambda^2 G^{(2)} + \lambda^4 G^{(4)})t]\, f_0\,, \tag{2.13}$$

$$y_4^\lambda(t) = [1 + \lambda^2 M^{(2)}(t)]\, x_4^\lambda(t)\,. \tag{2.14}$$

We say that the $A_{ij}(t)$ satisfy the $(\{c_n\}, \alpha)$ - <u>mixing condition</u> if there are positive constants $\{c_n : n = 0,2,\ldots\}$, α such that:

(i) the series $\sum_{n=0}^\infty c_{2n} z^n,\ \sum_{n=0}^\infty c_{2n+1} z^n$ have infinite radius of convergence,

(ii) $$\int\cdots\int_{u \geq v_1 \geq \cdots \geq v_n \geq s} \| A_{01}(u) A_{11}(v_1) \cdots A_{11}(v_n) R_m^\lambda(s) \|\, dv_1 \ldots dv_n \leq c_n (u-s)^{[n/2]} e^{-\alpha(u-s)},$$

for all $n = 0,1,\ldots$ where $[n/2]$ is the largest integer not exceeding n/2, and

$$R_m^\lambda(s) = AF^{(m)}(s) - \sum_{p=1}^m \sum_{q=m+1-p}^m \lambda^{p+q-m-1} F^{(p)}(s)\, G^{(q)}, \quad m = 2,4. \tag{2.15}$$

(iii) $\|K^{(2)}(t)\| \leq k_2 e^{-\alpha t}$, $\|K^{(4)}(t)\| \leq k_4 (t/\alpha) e^{-\alpha t}$, $t \geq 0$,

for some constants k_2, k_4.

The main result of the paper is the following

THEOREM 1 Let $A(t)$ satisfy conditions (*), (**), and let f_0 be in $\mathcal{B}_0$. Let $f^\lambda(t)$ be the unique continuous solution on $[0,\infty)$ of

$$\frac{d}{dt} f^\lambda(t) = \lambda A(t) f^\lambda(t)\ , \quad f^\lambda(0) = f_0\ ,$$

with λ in a bounded interval $[0,\Lambda]$. Then, if the $A_{ij}(t)$ satisfy the $(\{c_n\},\alpha)$ – mixing conditions, the integrals defining $G^{(2)}, G^{(4)}, M^{(2)}(t)$ exist, and there are positive functions β_2, β_4, bounded on compacts, such that

$$\| P_0 f^\lambda(t) - y_n^\lambda(t) \| \leq \lambda^n \beta_n(\lambda^2 t) \sup_{0\leq s\leq t} \| x_n^\lambda(s) \|\ , \quad n = 2, 4, \tag{2.16}$$

for all $t \geq 0$, λ in $[0,\Lambda]$. Moreover, the following bounds hold:

$$\left.\begin{aligned} &\lambda^2 \| G^{(2)} \| t \leq K_2 (\lambda/\alpha) \lambda t\ , \quad \lambda^4 \| G^{(4)} \| t \leq (K_2^2 + K_4)(\lambda/\alpha)^3 \lambda t\ , \\ &\lambda^2 \| M^{(2)}(\infty) \| \leq K_2 (\lambda/\alpha)^2\ , \quad \lambda^2 \| M^{(2)}(t) - M^{(2)}(\infty) \| \leq K_2 (\lambda/\alpha)^2 e^{-\alpha t}. \end{aligned}\right\} \tag{2.17}$$

THEOREM 2 With the hypotheses of Theorem 1, suppose in addition that c_{2n}, c_{2n+1} are bounded by $c_0 (K/\alpha)^n / n!$ with $K < (\alpha/\lambda)^2$. Then there are positive constants a_n, b_n and c such that $\beta_n(t) = a_n + b_n t$, $n = 2, 4$, and

$$\| P_0 f^\lambda(t) - x_4^\lambda(t) \| \leq \lambda^2 (c + b_4 \lambda^4 t) \sup_{0\leq s\leq t} \| x_4^\lambda(s) \| \tag{2.18}$$

for all t in $[0,\infty)$, λ in $[0,\Lambda]$.

The proofs of Theorems 1 and 2 will be given at the end of §3, after some technical results about differential equations. The boundedness of $A(t)$ is not strictly necessary, and the more general assumptions leading to the same result will also be formulated in §3.

The $(\{c_n\},\alpha)$ – mixing conditions may seem very contrived; in §4 we shall present a class of models for which it is satisfied, including the examples mentioned in §1. Applications to the reduced dynamics of quantum systems coupled to boson or fermion reservoirs will be given in a separate paper [5]; see also [11].

We comment briefly on our results, showing how the well-known weak coupling limit theory [1, 2, 3] can be recovered from our estimate (2.16) for $n = 2$ and how corrections to exponential behaviour are needed in order to improve the approximation on time intervals $0 \leq t \leq \tau/\lambda^2$. Our estimate (2.16) becomes

$$\| P_0 f^\lambda(t) - \exp[\lambda^2 t G^{(2)}] f_0 \| \leq \lambda^2 \beta_2(\lambda^2 t) \sup_{0\leq s\leq t} \| \exp[\lambda^2 G^{(2)} s] f_0 \| \tag{2.19}$$

for $n = 2$ and

$$\| P_0 f^\lambda(t) - (1+\lambda^2 M^{(2)}(t)) \exp[\lambda^2 t (G^{(2)} + \lambda^2 G^{(4)})] f_0 \| \leq \lambda^4 \beta_4(\lambda^2 t) \sup_{0\leq s\leq t} \| \exp[\lambda^2 s (G^{(2)} + \lambda^2 G^{(4)})] f_0 \| \quad (2.20)$$

for $n = 4$; our estimate (2.18) is

$$\| P_0 f^\lambda(t) - \exp[\lambda^2 t (G^{(2)} + \lambda^2 G^{(4)})] f_0 \| \leq \lambda^2 (c + b_4 \lambda^4 t) \sup_{0\leq s\leq t} \| \exp[\lambda^2 s (G^{(2)} + \lambda^2 G^{(4)})] f_0 \| . \quad (2.21)$$

Put $\tau = \lambda^2 t$, $\sigma = \lambda^2 s$ in (2.19). Then for $0 \leq \tau < \infty$, $0 \leq \lambda \leq \Lambda$, we have

$$\| P_0 f^\lambda(\tau/\lambda^2) - \exp[\tau G^{(2)}] f_0 \| \leq \lambda^2 \beta_2(\tau) \sup_{0\leq\sigma\leq\tau} \| \exp[\sigma G^{(2)}] f_0 \| ,$$

and, taking the limit $\lambda \to 0$,

$$\lim_{\lambda\to 0} \| P_0 f^\lambda(\tau/\lambda^2) - \exp[\tau G^{(2)}] f_0 \| = 0 \quad (2.22)$$

uniformly on all compact intervals $0 \leq \tau \leq \tau_1 < \infty$. In particular, it follows from (2.22) and assumption (*) that $\{\exp[\tau G^{(2)}] : \tau \geq 0\}$ is a semigroup of contractions.

The estimate (2.21) proves that $\exp[\lambda^2 t (G^{(2)} + \lambda^2 G^{(4)})] f_0$ is an approximation to $P_0 f^\lambda(t)$ for longer times than $\exp[\lambda^2 t G^{(2)}] f_0$. If $\{\exp[\tau(G^{(2)} + \lambda^2 G^{(4)})] : \tau \geq 0\}$ is a semigroup of contractions for λ sufficiently small, then it follows from (2.21) that

$$\lim_{\lambda\to 0} \sup_{0\leq \lambda^4 t \leq \tau_1} \| P_0 f^\lambda(t) - \exp[\lambda^2 t (G^{(2)} + \lambda^2 G^{(4)})] f_0 \| = 0 \quad (2.23)$$

for all $0 \leq \tau_1 < \infty$. Estimates of the from (2.23) have been obtained by Palmer [12] for a class of models for which $A_{11}(t)$ vanishes identically (and $A_{00}(t)$ need not).

The estimate (2.20) tells us that $(1 + \lambda^2 M^{(2)}(t)) \exp[\lambda^2 t (G^{(2)} + \lambda^2 G^{(4)})] f_0$ is a closer approximation to $P_0 f^\lambda(t)$ than $\exp[\lambda^2 t G^{(2)}] f_0$ on the same time scale, for λ sufficiently small. For t large in comparison to $1/\alpha$, this expression has essentially an exponential behaviour, since $M^{(2)}(t)$ practically reaches its limit value $M^{(2)}(\infty)$, so it essentially contains a short-time correction to exponential decay. This is needed in order to improve the approximation for small as well as for large times. Indeed, we have

$$\frac{d}{dt} P_0 f^\lambda(t) \Big|_{t=0} = 0 \qquad \text{for all } f_0 \text{ in } \mathcal{B}_0 ,$$

since $A_{00}(0) = 0$ by assumption; and $\frac{d}{dt} x_n^\lambda(t) \Big|_{t=0}$, $n = 2, 4$, is of order λ^2 in general. But

$$\frac{d}{dt} y_4^\lambda(t) \Big|_{t=0} = [\lambda^2 (G^{(2)} + \dot{M}^{(2)}(0)) + \lambda^4 G^{(4)}] f_0 = \lambda^4 G^{(4)} f_0$$

of order λ^4, since $G^{(2)} + \dot{M}^{(2)}(0) = P_0 A(0) F^{(1)}(0) = 0$.

Finally, we point out that (2.16), (2.17) indicate that an asymptotic series in (λ/α) emerges. In contrast to the equations of classical mechanics to which the averaging method is applied (such as the van der Pol equation $\ddot{x} - \varepsilon(1 - x^2)\dot{x} + \omega^2 x = 0$), the characteristic times $1/\lambda$ and $1/\alpha$ are not both parameters in the original equation; the characteristic time $1/\alpha$, which describes the decay of correlations in the reservoir, appears only when the averaging operation is performed, and it is only at the very end that the asymptotic series in (λ/α) emerges. In this respect the averaging method applied to stochastic evolution equations is more obscure than when applied to equations of classical mechanics.

§3. TECHNICAL RESULTS AND PROOFS OF THE THEOREMS:

We consider the differential equation (2.1) whose solution is given by (2.2), (2.3). If f_0 is in $\mathcal{B}_0$ we want to approximate $P_0 f^\lambda(t)$ to order n in λ by an expression of the form

$$y_n^\lambda(t) = [1 + M_{n-1}^\lambda(t)]\, x_n^\lambda(t), \tag{3.1}$$

with

$$x_n^\lambda(t) = \exp[G_n^\lambda t] f_0 \quad , \tag{3.2}$$

where

$$G_n^\lambda = \sum_{r=1}^{n} \lambda^r G^{(r)}, \quad M_{n-1}^\lambda(t) = \sum_{r=1}^{n-1} \lambda^r M^{(r)}(t), \tag{3.3}$$

and $G^{(r)}$, $M^{(r)}(t)$ are in $\mathcal{L}(\mathcal{B}_0)$, with $M^{(r)}(0) = 0$ for all r. To do so, we introduce an auxiliary expression

$$g_n^\lambda(t) = (1 + F_n^\lambda(t))\, x_n^\lambda(t), \tag{3.4}$$

where

$$F_n^\lambda(t) = \sum_{r=1}^{n} \lambda^r F^{(r)}(t), \tag{3.5}$$

and the $F^{(r)}(t)$ are in $\mathcal{L}(\mathcal{B}_0, \mathcal{B})$, such that $F^{(r)}(0) = 0$ and

$$\frac{d}{dt} F^{(r)}(t) f_0 = \dot{F}^{(r)}(t) f_0 \tag{3.6}$$

exists and is a continuous function of t for all f_0 in $\mathcal{B}_0$, $r = 1, \dots n$. Then we set

$$M^{(r)}(t) = P_0 F^{(r)}(t), \quad r = 1, \dots, n, \tag{3.7}$$

so that

$$y_n^\lambda(t) = P_0 g_n^\lambda(t) - P_0 F^{(n)}(t)\, x_n^\lambda(t). \tag{3.8}$$

THEOREM 3. Suppose that $G^{(r)}, F^{(r)}(t)$, $r = 1, \dots, n$, satisfy the hierarchy of equations

$$\left.\begin{aligned} G^{(1)} + \dot{F}^{(1)}(t) &= A(t) P_0 , \\ G^{(r)} + \dot{F}^{(r)}(t) &= A(t) F^{(r-1)}(t) - \sum_{s=1}^{r-1} F^{(r-s)}(t)\, G^{(s)}, \quad r = 2,3,\dots,n. \end{aligned}\right\} \tag{3.9}$$

Then there are constants $a_n(T)$, $b_n(\Lambda, T)$ such that

$$\| P_0 f^\lambda(t) - y_n^\lambda(t) \| \le \lambda^n [\, a_n(T) + b_n(\Lambda, T)\,] \sup_{0 \le s \le t} \| x_n^\lambda(s) \| \tag{3.10}$$

for all $0 < \lambda \le \Lambda$, $0 \le t \le T$, for all f_0 in $\mathcal{B}_0$. Explicity, we have

$$a_n(T) = \sup_{0 \le t < T} \| P_0 F^{(n)}(t) \| , \tag{3.11}$$

$$b_n(\Lambda, T) = \sup_{0 < \lambda \le \Lambda,\, 0 \le t \le T} \lambda \int_0^t \| P_0 U^\lambda(t,s) R_n^\lambda(s) \| \, ds, \tag{3.12}$$

where

$$R_n^\lambda(s) = A(s)F^{(n)}(s) - \sum_{p=1}^{n}\sum_{q=n+1-p}^{n} \lambda^{p+q-n-1} F^{(p)}(s)G^{(q)}. \tag{3.13}$$

PROOF By definition of $y_n^\lambda(t)$, $g_n^\lambda(t)$, it suffices to prove that $\|P_o[f^\lambda(t) - g_n^\lambda(t)]\| \leqslant \lambda^n b_n(\Lambda,T)$. We adapt the well-known method for the approximation of semigroups (cf [13] Chapter IX §2) The expression $U^\lambda(t,s)[1+F_n^\lambda(s)]\exp[G_n^\lambda s]f_o$ equals $f^\lambda(t)$ for $s=0$ and $g_n^\lambda(t)$ for $s=t$. Therefore

$$f^\lambda(t) - g_n^\lambda(t) = -\int_0^t \frac{d}{ds}\{U^\lambda(t,s)[1+F_n^\lambda(s)]\exp[G_n^\lambda s]f_o\}\, ds$$
$$= \int_0^t U^\lambda(t,s)\{\lambda A(s)[1+F_n^\lambda(s)] - [1+F_n^\lambda(s)]G_n^\lambda - \dot F_n^\lambda(s)\}x_n^\lambda(s)\, ds.$$

Using the hierarchy of equations (3.9) and the definition (3.13) of $R_n^\lambda(s)$ we get

$$f^\lambda(t) - g_n^\lambda(t) = \lambda\int_0^t U^\lambda(t,s)R_n^\lambda(s)x_n^\lambda(s)\, ds$$

from which the conclusion follows by projecting onto $\mathcal{B}_o$, taking norms, and obvious majorizations.

The solution of the hierarchy of equations (3.9) is not unique. We determine it by introducing an averaging operations.

Let $\mathcal{A}$ be a linear space consisting of strongly continuous functions $t \mapsto A(t)$ on $\mathbb{R}_+$ with values in $\mathcal{L}(\mathcal{B})$ (not necessarily all of them), containing 1 and such that $\mathcal{A}\mathcal{L}(\mathcal{B}_o)$ and $\mathcal{L}(\mathcal{B}_o)\mathcal{A}$ are contained in $\mathcal{A}$. An average on $\mathcal{A}$ is a linear map $\mathcal{E}$ of $\mathcal{A}$ onto $\mathcal{L}(\mathcal{B}_o)$ such that $\mathcal{E}(1) = 1$ and

$$\mathcal{E}(A\,\mathcal{E}(B)) = \mathcal{E}(A)\mathcal{E}(B) = \mathcal{E}(\mathcal{E}(A)B), \tag{3.14}$$

for all A, B in $\mathcal{A}$. Notice that $\mathcal{L}(\mathcal{B}_o)$ can be identified with

the subspace of $\mathcal{A}$ consisting of the constant functions with values in $\mathcal{L}(\mathcal{B}_o)$, then it follows from the definition that $\mathcal{E}$ leaves $\mathcal{L}(\mathcal{B}_o)$ pointwise invariant.

We introduce the notation

$$(\eta A)(t) = \int_0^t [A(s) - \mathcal{E}(A)]\, ds \tag{3.15}$$

for A in $\mathcal{A}$, $t \geqslant 0$.

THEOREM 4. Let $\mathcal{E}$ be an average. The hierarchy of equations (3.9) has a unique solution subject to the conditions

$$G^{(r)} = \mathcal{E}(G^{(r)}), \quad \mathcal{E}(\dot F^{(r)}) = 0, \quad r = 1,\dots,n, \tag{3.16}$$

and satisfying $F^{(r)}_{(0)} = 0$ for all r. It is given recursively by

$$G^{(r)} = \mathcal{E}(H^{(r)}), \quad F^{(r)} = \eta(H^{(r)}), \quad r = 1,\dots,n, \tag{3.17}$$

where $H^{(1)} = AP_o$ and

$$H^{(r)} = A\eta H^{(r-1)} - \sum_{s=1}^{r} \eta(H^{(r-s)})\mathcal{E}(H^{(s)}), \quad r = 2,\dots,n, \tag{3.18}$$

provided all $H^{(r)}$, $r = 1,\dots,n$, are in $\mathcal{A}$.

PROOF Given H in $\mathcal{A}$, the unique solution of the equation $G+\dot{F}=H$, $F(0)=0$, such that $\mathcal{E}(G)=G$, $\mathcal{E}\dot{F}=0$, is $G=\mathcal{E}H$, $F=\eta H$. If $H^{(1)},\dots,H^{(r-1)}$ are in $\mathcal{A}$ then $H^{(r)}$ is determined by (3.18) and (3.17) makes sense if $H^{(r)}$ is in $\mathcal{A}$.

The meaning of the condition $\mathcal{E}(G^{(r)}) = G^{(r)}$ is just that the $G^{(r)}$ are time independent operators in $\mathcal{L}(\mathcal{B}_0)$. We choose $\mathcal{E}$ in such a way that the condition $\mathcal{E}(\dot{F}^{(r)})=0$ becomes a condition of slow growth for the functions $M^{(r)}(t)=P_0F^{(r)}(t)$.

Define

$$\mathcal{E}(A) = \lim_{T\to\infty} \frac{1}{T}\int_0^T P_0 A(t) P_0 \, dt \tag{3.19}$$

when this limit exists. Then $\mathcal{E}(\dot{F}^{(r)})=0$, $F^{(r)}(0)=0$ gives

$$\lim_{t\to\infty} \frac{1}{t} \| P_0 F^{(r)}(t) \| = 0. \tag{3.20}$$

Actually, our assumption of $(\{C_n\},\alpha)$ – mixing will ensure that $P_0F^{(2)}(t)$, $P_0F^{(4)}(t)$ are uniformly bounded on $[0,\infty)$ *if* the averaging operation (3.19) is used.

Under assumptions (2.5) - (2.7) and $(\{C_n\},\alpha)$ – mixing, the expressions $P_0H^{(r)}(t)P_0$ will be shown to have a limit as $t\to\infty$, so that $\mathcal{E}(H^{(r)}) = \lim_{t\to\infty} P_0H^{(r)}(t)P_0$, $r = 2,4$. In [5], assumptions (2.5) - (2.7) will be relaxed somewhat and the use of the time average (3.19) instead of the limit as $t\to\infty$ will be essential.

PROOF OF THEOREM 1 We have by (2.5), (2.7), (2.8), (3.18)

$$P_0H^{(2)}(t)P_0 = -\int_0^t K^{(2)}(s)\,ds \tag{3.21}$$

which tends to a limit as $t\to\infty$ by assumption (iii). Then this limit is $G^{(2)}$, and it is given by (2.9). We have $M^{(1)}\equiv 0$ by (**), and

$$M^{(2)}(t) = \int_0^t\int_s^\infty K^{(2)}(u)\,du$$

which becomes (2.11), with the same change of variables. The bounds (2.17) concerning $G^{(2)}$ and $M^{(2)}$ follow immediately from (iii). This proves in particular that $\|P_0F^{(2)}(t)\|$ is bounded uniformly in t. Using equation (2.6), (2.7), (2.8), (3.18), we find

$$P_0H^{(4)}(t)P_0 = \int_0^t K^{(4)}(s)\,ds - \int_0^t K^{(2)}(t-s)M^{(2)}(s)\,ds + \left\{\int_0^t\int_0^s K^{(2)}(t-u)\,du\,ds - M^{(2)}(t)\right\}G^{(2)}. \tag{3.22}$$

We prove that this expression has a limit as $t\to\infty$ which is then $G^{(4)}$. The first term tends to the first term of (2.10), and the second term is $-\int_0^t K^{(2)}(s)M^{(2)}(t-s)\,ds$ which tends to $-\int_0^\infty K^{(2)}(s)M^{(2)}(\infty)\,ds$, since $M^{(2)}(t)$ approaches its limit exponentially fast. The third term is

$$\left\{\int_0^t K^{(2)}(t-u)(t-u)\,du - M^{(2)}(t)\right\}G^{(2)} = t\int_t^\infty K^{(2)}(s)\,ds\; G^{(2)},$$

which vanishes in the limit $t\to\infty$, by assumption (iii). Then

$G^{(4)}$ is given by (2.10), and the bound on $G^{(4)}$ in (2.17) follows. It is also easily seen that $P_o H^{(4)}_{(t)} P_o - G^{(4)}$ is bounded by an exponentially decreasing function of t, so that $\| P_o F^{(4)}_{(t)} \|$ is bounded uniformly in t.

Now we use Theorem 3. We have just found that

$a_4 \equiv \sup_{0 \le T < \infty} a_4(T)$ is finite. We have also

$$b_4(\Lambda, T) \le \sup_{0<\lambda\le\Lambda,\ 0\le t\le T} \lambda \Big[\int_0^t \| P_o U^\lambda_{(t,s)} P_o R^\lambda_4(s) \| ds + \int_0^t \| P_o U^\lambda_{(t,s)} P_1 R^\lambda_4(s) \| ds \Big] .$$

Now

$$\| P_o U^\lambda_{(t,s)} P_o R^\lambda_4(s) \| \le \| P_o R^\lambda_4(s) \| \le \sum_{p=1}^{2} \sum_{q=2-p}^{2} \lambda^{2(p+q)-3} \| P_o F^{(2p)}_{(s)} G^{(2q)} \|$$

which is uniformly bounded and of order λ . So there is a constant $\tilde{b}_4$ such that

$$\lambda \int_0^t \| P_o U^\lambda(t,s) P_o R^\lambda_4(s) \| ds \le \tilde{b}_4 \lambda^2 t .$$

We also have

$$P_o U^\lambda_{(t,s)} P_1 - \lambda \int_s^t P_o U^\lambda_{(t,u)} P_o A_{01}(u) du$$

$$= \sum_{n=1}^{\infty} \lambda^n \int \cdots \int_{s \le u_n \le \cdots \le u_1 \le t} P_o A(u_1) \cdots A(u_{n-1}) A_{11}(u_n) du_n \cdots du_1 .$$

$$= \lambda \int_s^t P_o U^\lambda_{(t,u)} P_1 A_{11}(u) du .$$

By iteration we obtain

$$P_o U^\lambda_{(t,s)} P_1 = \sum_{n=0}^{\infty} \lambda^{n+1} \int_{u=s}^{t} \int \cdots \int_{s \le v_n \le \cdots \le v_1 \le u} P_o U^\lambda_{(t,u)} A_{01}(u) A_{11}(v_1) \cdots A_{11}(v_n) dv_n \ldots dv_1 du \qquad (3.23)$$

(cf. Theorem 3.1 of [1], II). Now if we use assumption (ii), we find

$$\lambda \int_0^t \| P_o U^\lambda_{(t,s)} P_1 R^\lambda_4(s) \| ds \le \lambda^2 \sum_{n=0}^{\infty} \iint_{0 \le u \le s \le t} \lambda^n c_n (u-s)^{[n/2]} e^{-\alpha|u-s|} du ds . \qquad (3.24)$$

Since $|u-s|^2 e^{-\alpha|u-s|}$ is bounded and $\int_0^t s^k ds < t^{k+1}$, we have

$$|(3.24)| \le (\text{const.}) \Lambda^2 \sum_{n=0}^{\infty} [c_{2n} (\lambda^2 t)^n + \Lambda^2 c_{2n+1} (\lambda^2 t)^n] \equiv \tilde{\beta}_4(\lambda^2 t) .$$

By assumption (i) on the coefficients $\{c_n\}$, $\tilde{\beta}_4(\lambda^2 t)$ is bounded on compacts. Then the estimate (2.16) holds with

$\beta_4(t) = a_4 + \tilde{b}_4 t + \tilde{\beta}_4(t)$, for $n = 4$.

The proof for $n = 2$ has the same structure.

<u>PROOF OF THEOREM 2</u> If $c_{2n}, c_{2n+1} \le c_o (\kappa/\alpha)^n / n!$ with $\kappa < (\alpha/\lambda)^2$, we can bound (3.24) by

$$(1+\Lambda^2) \lambda^2 c_o \sum_{n=0}^{\infty} \frac{1}{n!} \iint_{0 \le u \le s \le t} [\tfrac{\lambda^2 \kappa}{\alpha}(u-s)]^n e^{-\alpha(u-s)} du ds$$

$$= (1+\Lambda^2) \lambda^2 c_o \iint_{0 \le u \le s \le t} \exp\Big[(\tfrac{\lambda^2 \kappa}{\alpha} - \alpha)(u-s) \Big] du ds$$

and since the exponential is decreasing this is bounded by $[2c_o(\alpha - \frac{\lambda^2\kappa}{\alpha})^{-1}] \lambda^2 t$. Then the linear bound $\beta_4(t) = a_4 + b_4 t$ follows with $b_4 = \tilde{b}_4 + 2c_o(\alpha - \frac{\lambda^2\kappa}{\alpha})^{-1}$. Again, the proof for $n = 2$ has the same structure. Moreover, for $n = 4$ we have

$$\| P_o f^\lambda(t) - x^\lambda_4(t) \| \le \| P_o f^\lambda(t) - y^\lambda_4(t) \| + \lambda^2 \sup_{0\le s\le\infty} \| M^{(2)}(s) \| \sup_{0 \le s \le t} \| x^\lambda_4(s) \|$$

$$\le \lambda^2 (a_4 + b_4 \lambda^2 t + \sup_{0\le s\le\infty} \| M^{(2)}(s) \|) \sup_{0\le s\le t} \| x^\lambda_4(s) \| ,$$

which is the bound (2.18), with $c = a_4 + \sup_{0\le s\le\infty} \| M^{(2)}(s) \|$.

REMARK A The boundedness of $A(t)$ is not strictly needed. Theorems 1 - 4 above hold for the approximation of

$P_o f^\lambda(t) = P_o U^\lambda(t,o) f_o$, f_o in $\mathcal{B}_o$, whenever one is given a two-parameter family of propagators $\{U(t,s) : o \leq s \leq t\}$ on $\mathcal{B}$ and a domain $\mathcal{D}$ in $\mathcal{B}$, containing $\mathcal{B}_o$, such that

$\frac{d}{ds} U^\lambda(t,s) f = -U^\lambda(t,s) A(s) f$ for all f in $\mathcal{D}$, $o \leq s \leq t$,

$A(s)$ in a linear map of $\mathcal{D}$ into itself for all $s \geq o$,

$s \mapsto A(s) f$ is a continuous function of $s \in \mathbb{R}_+$ for all f in $\mathcal{D}$, $\int \cdots \int_{I_n} A(s_1) \cdots A(s_n) f \, ds_1 \cdots ds_n$ is in $\mathcal{D}$ for all f in $\mathcal{D}$, for any compact measurable subset I_n of $\mathbb{R}_+^n$ and for all n, and assumptions (*), (**) hold -

REMARK B It is also possible to consider the case in which $\lambda A(t)$ is given by a power series

$$\lambda A(t) = \sum_{r=1}^{\infty} \lambda^r A^{(r)}(t) , \tag{3.25}$$

with $\sum_{r=1}^{\infty} \lambda^r \sup_{0 \leq t < \infty} \| A^{(r)}(t) \| < \infty$ for $\lambda \leq \Lambda$. Then the hierarchy of equations becomes

$$G^{(r)} + \dot{F}^{(r)} = A^{(r)} + \sum_{s=1}^{r-1} (A^{(s)} F^{(r-s)} - F^{(r-s)} G^{(s)}) \tag{3.26}$$

and its solution can be determined with the aid of an average, as in Theorem 4. The estimate of Theorem 3 becomes

$$\| P_o f^\lambda(t) - y_n^\lambda(t) \| \leq \lambda^n [a_n(T) + b_n(\Lambda,T) + c_n(\Lambda,T)] \sup_{o \leq s \leq t} \| x_n^\lambda(s) \| , \tag{3.27}$$

where

$$c_n(\Lambda,T) = \Lambda T \Big[\sup_{0 < \lambda \leq \Lambda} \sup_{0 \leq s \leq t \leq T} \| U^\lambda(t,s) \| \sum_{p=n+1}^{\infty} \Lambda^{p-n-1} \sup_{0 \leq t \leq T} \| A^{(r)}(t) \| \Big] \tag{3.28}$$

and

$$R_n^\lambda = \sum_{p=1}^{n} \sum_{q=n+1-p}^{n} \lambda^{p+q-n-1} [A^{(q)} F^{(p)} - F^{(p)} G^{(q)}] . \tag{3.29}$$

§4. AN APPLICATION TO STOCHASTIC DIFFERENTIAL EQUATIONS

Let $\mathcal{B}_o$ be a finite-dimensional Hilbert space, and let (Ω,μ) be a probability space. Let $\mathcal{B}$ be the Hilbert space $L^2(\Omega,\mu;\mathcal{B}_o) = \mathcal{B}_o \otimes L^2(\Omega,\mu)$; $\mathcal{B}_o$ may be identified with $\mathcal{B}_o \otimes 1$. Let X_i , $i=1,\dots,r$, be skew-adjoint operators on $\mathcal{B}_o$ and let $\phi_i(t)$, $t\in\mathbb{R}$, $i=1,\dots,r$, be real-valued Gaussian random variables on (Ω,μ), with mean zero and covariance matrix

$$\langle\phi_i(s)\phi_j(t)\rangle = k_{ij}(t-s) , \tag{4.1}$$

and consider the family $\{A(t) : t\in\mathbb{R}\}$ of skew-adjoint operators in $\mathcal{B}$ given by

$$A(t) = \sum_{i=1}^{r} X_i \otimes \phi_i(t) , \quad t \in \mathbb{R} . \tag{4.2}$$

Then it is possible to make sense of $U^\lambda(t,s) = T\exp[\lambda\int_s^t A(u)\,du]$ even if $A(t)$ is unbounded, and the assumptions of Remark A are satisfied; $\{U^\lambda(t,s) : 0\leqslant s\leqslant t\}$ is a family of unitaries, so that condition (*) holds, and condition (**) holds because $\phi_i(t)$ are zero-mean Gaussian variables.

THEOREM 5 If there are positive constants K and α such that

$$|\langle\phi_i(s)\phi_j(t)\rangle| \leqslant K\exp[-\alpha|t-s|] , \quad i,j = 1,\dots,r, \; t,s\in\mathbb{R}, \tag{4.3}$$

then the $(\{c_n\},\alpha)$ – mixing condition holds, with

$$c_{2n}, c_{2n+1} \leqslant c_0(\varepsilon) \,[\,2Kr^2\|X\|^2(1+\varepsilon)\alpha^{-1}]^n/n! \;, \tag{4.4}$$

where $\varepsilon>0$ and $\|X\| = \sup\{\|X_i\| ; i=1,\dots,r\}$. Then the estimates of Theorem 1 (and of Theorem 2 for λ small enough) follow.

PROOF We have

$$K^{(2)}(t_1-t_2) = -\sum_{i,k=1}^{r} X_i X_k \, k_{ik}(t_1-t_2) , \tag{4.5}$$

$$\begin{aligned} K^{(4)}(t_1-t_4) = \sum_{i,j,k,l=1}^{r} \{ & X_iX_jX_kX_l \iint_{t_1\geqslant t_2\geqslant t_3\geqslant t_4} k_{ik}(t_1-t_3)k_{jl}(t_2-t_4)\,dt_3\,dt_2 \\ & + X_iX_jX_lX_k \iint_{t_1\geqslant t_2\geqslant t_3\geqslant t_4} k_{ik}(t_1-t_4)k_{jl}(t_2-t_3)\,dt_3\,dt_2\} , \end{aligned} \tag{4.6}$$

so that the estimates (iii) hold, with $K_2 \leqslant Kr^2\|X\|^2$, $K_4 \leqslant 8K^2r^4\|X\|^4$. Now we have to prove the estimates (ii) involving R_2^λ and R_4^λ , with constants $\{c_n\}$ satisfying (4.4), so that also (i) holds. We give the argument for R_4^λ , for R_2^λ the procedure would be similar but easier. We have

$$P_1 R_4^\lambda(s) = A_{10}(s) P_0 F^{(4)}(s) + A_{11}(s) P_1 F^{(4)}(s) - [P_1 F^{(1)}(s) + \lambda P_1 F^{(2)}(s)]\, G^{(4)}$$

$$- [P_1 F^{(3)}(s) + \lambda P_1 F^{(4)}(s)]\,[G^{(2)} + \lambda^2 G^{(4)}] .$$

Since $\| P_0 F^{(4)}(s) \|$ is uniformly bounded by Theorem 1, we have to prove estimates of the form (ii), where $P_1 R_4^\lambda(s)$ is replaced in turn by the following quantities:

$$A_{10}(s) \; ; \; P_1 F^{(1)}(s) = \int_0^s A_{10}(u)\, du \; ; \; P_1 F^{(2)}(s) = \int_0^s A_{11}(u) P_1 F^{(1)}(u)\, du \; ;$$

$$P_1 F^{(3)}(s) = \int_0^s A_{10}(u) P_0 F^{(2)}(u)\, du + \int_0^s \{ A_{11}(u) P_1 F^{(2)}(u) - P_1 F^{(1)}(u) G^{(2)} \}\, du$$

(the first term on the right-hand side gives a contribution of the same kind as $P_1 F^{(1)}(s)$);

$$P_1 F^{(4)}(s) = \int_0^s \{ A_{11}(u) P_1 F^{(3)}(u)\, du - P_1 F^{(2)}(u) G^{(2)} \}\, du \; ; \quad A_{11}(s) P_1 F^{(4)}(s) .$$

When a quantity containing an odd number of A's is inserted, only the contributions to (ii) with n even survive, and vice versa. The study of the single contributions is essentially the same for all, so we only give the details for one of them. Take for instance

$$\int_0^s A_{11}(u) P_1 F^{(2)}(u)\, du = \iiint\limits_{0 \le w_3 \le w_2 \le w_1 \le s} A_{11}(w_1) A_{11}(w_2) A_{10}(w_3)\, dw_3\, dw_2\, dw_1 \; ;$$

then we must show that, for each n, the expression

$$\int\limits_{0 \le w_3 \le \dots \le w_1 \le s} \!\!\cdots\!\! \int \; \int\limits_{s \le v_{2n} \le \dots \le v_1 \le u} \!\!\cdots\!\! \int \| A_{01}(u) A_{11}(v_1) \dots A_{11}(v_{2n}) A_{11}(w_1) \cdots A_{11}(w_2) A_{10}(w_3) \| \, dw_1 \dots dw_3\, dv_1 \dots dv_{2n} \quad (4.7)$$

is bounded by $c_{2n} (u-s)^n \exp[-\alpha |u-s|]$, where c_{2n} satisfies the bounds (4.4).

For all (2n + 2) -tuples $(t_0, t_1, \dots, t_{2n+1})$, let

$$I(t_0, \dots, t_{2n+1}) = \| A_{01}(t_0) A_{11}(t_1) \dots A_{11}(t_{2n}) A_{10}(t_{2n+1}) \| .$$

We prove in the Appendix that for each $\varepsilon > 0$ there exists a constant $K_0(\varepsilon)$ such that, for $t_0 \ge t_1 \ge \dots \ge t_{2n} \ge t_{2n+1}$,

$$I(t_0, \dots, t_{2n+1}) \le K_0(\varepsilon) (2 \kappa r^2 \|X\|^2 (1+\varepsilon))^{n+1} \exp[-\alpha(t_0 + t_{2n-1} - t_{2n} - t_{2n+1})] \sum_{p \in \mathcal{P}_{n-1}} \prod_{q=1}^{n-1} \exp[-\alpha(t_{p(2q-1)} - t_{p(2q)})]$$

where $\mathcal{P}_{n-1}$ is the set of those permutations p of $\{1, \dots, 2n-2\}$ such that $p(2q-1) < p(2q)$, $p(2q-1) < p(2q+1)$ for all q.

Then let $t_0 \ge \dots \ge t_{2n+3}$ be the time variables $u \ge v_1 \ge \dots \ge v_{2n} \ge w_1 \ge \dots \ge w_3$ appearing in the integrand of (4.7). Applying the estimate (4.8) gives

$$|(4.7)| \le K_0(\varepsilon) (2 \kappa r^2 \|X\|^2 (1+\varepsilon))^{n+2} \exp[-\alpha u] \int\limits_{0 \le w_3 \le \dots \le w_1 \le s} \!\!\cdots\!\! \int \exp[-\alpha(w_1 - w_2 - w_3)]\, dw_1\, dw_2$$

$$\times \int\limits_{s \le v_{2n} \le \dots \le v_1 \le u} \!\!\cdots\!\! \int \; \sum_{p \in \mathcal{P}_n} \prod_{q=1}^{n} \exp[-\alpha(v_{p(2q-1)} - v_{p(2q)})]\, dv_1 \dots dv_{2n} . \quad (4.9)$$

By elementary integration, we have

$$\int\limits_{0 \le w_3 \le \dots \le w_1 \le s} \!\!\cdots\!\! \int \exp[-\alpha(w_1 - w_2 - w_3)]\, dw_1\, dw_2\, dw_3 \le (\text{const.}) \exp[\alpha s] \quad (4.10$$

and by a Lemma of Davies (Lemma 3.3 of [1], I, cf. also Pulè [14] Lemma 3)

$$\int\cdots\int_{s\leq v_{2n}\leq\cdots\leq v_1\leq u} \sum_{p\in\mathcal{P}_n} \prod_{q=1}^{n} \exp[-\alpha(v_{p(2q-1)} - v_{p(2q)})] \leq (u-s)^n/\alpha^n n! \quad . \tag{4.11}$$

Put $c_0(\varepsilon) = (\text{Const.})K_0(\varepsilon)(2\kappa r^2\|X\|^2(1+\varepsilon))^2$ and insert (4.10) and (4.11) in (4.9). The result is

$$|(4.7)| \leq c_0(\varepsilon)[2\kappa r^2\|X\|^2(1+\varepsilon)\alpha^{-1}]^n[(u-s)^n/n!]\exp[-\alpha(u-s)] \quad ,$$

which is an estimate of the form (ii), with coefficients satisfying (4.4).

Similar arguments apply to the other contributions to $P_1R_4^\lambda$. Indeed, by the same method as used in the Appendix, it is possible to prove that for any fixed integer $\overline{m}$, and for any $m=0,\dots,\overline{m}$,

$$I(t_0,\dots,t_{2n+1}) \leq K_0(\varepsilon,\overline{m})(2\kappa r^2\|X\|^2(1+\varepsilon))^2$$
$$\times \exp[-\alpha t_0 - \alpha\sum_{j=0}^{m}(t_{2(n-j)-1} - t_{2(n-j)}) + \alpha t_{2n+1}]$$
$$\times \sum_{p\in\mathcal{P}_{n-1-m}} \prod_{q=1}^{n-1-m} \exp[-\alpha(t_{p(2q-1)} - t_{p(2q)})] \quad .$$

Then, for the contributions to $P_1R_4^\lambda(s)$ containing an odd number $2k+1$ of A's, the signs in the exponent are re-arranged to produce $\exp[-\alpha(w_1-w_2) - \alpha(w_3-w_4) - \dots + \alpha(w_{2k}+w_{2k+1})]$, for those containing an even number $2k$ of A's, the signs are re-arranged to give $\exp[-\alpha(v_{2n+1}-w_1) - \alpha(w_2-w_3) - \dots + \alpha(w_{2k-1}+w_{2k})]$ which is bounded above by $\exp[-\alpha(s-w_1) - \alpha(w_2-w_3) - \dots + \alpha(w_{2k-1}+w_{2k})]$ since $v_{2n+1} \geq s$. In both cases, integration over all variables w gives a bound of the form (4.10), and integration over all variables $v_1,\dots,v_{2n}$ can be extended to the domain $s\leq v_{2n}\leq\dots\leq v_1\leq u$ giving the bound (4.11).

This completes the proof of the estimate (ii) for $P_1R_4^\lambda(s)$.

For $P_1R_2^\lambda(s)$ a similar discussion applies, it is easier since a smaller number of contributions is involved.

<u>THEOREM 6</u> The explicit form of $G^{(2)}, G^{(4)}, M^{(2)}(t)$ is

$$G^{(2)} = \sum_{i,k=1}^{r} X_iX_k\int_0^\infty k_{ik}(t)\,dt \quad , \tag{4.12}$$

$$G^{(4)} = \sum_{i,j,k,l=1}^{r}\{(X_iX_jX_kX_l - X_jX_lX_iX_k)\lim_{t_1\to\infty}\iiint_{t_1\geq\dots\geq t_4\geq 0} k_{ik}(t_1-t_3)k_{jl}(t_2-t_4)\,dt_2\,dt_3\,dt_4$$
$$+(X_iX_jX_lX_k - X_jX_lX_iX_k)\lim_{t_1\to\infty}\iiint_{t_1\geq\dots\geq t_4\geq 0} k_{ik}(t_1-t_4)k_{jl}(t_2-t_3)\,dt_2\,dt_3\,dt_4\}, \tag{4.13}$$

$$M^{(2)}(t) = -\sum_{i,k=1}^{r} X_iX_k\,\{\int_0^\infty k_{ik}(s)\,s\,ds + \int_t^\infty (t-s)k_{ik}(s)\,ds\} \quad . \tag{4.14}$$

<u>PROOF</u> This is a straightforward consequence of equations (2.9), (2.10), (2.11), (4.5), (4.6), and of the identity:

$$\int_0^\infty f(t)\,dt\int_0^\infty g(s)s\,ds = \lim_{t_1\to\infty}\iiint_{t_1\geq\dots\geq t_4\geq 0}[g(t_1-t_3)f(t_2-t_4)+g(t_1-t_4)f(t_2-t_3)]\,dt_2\,dt_3\,dt_4 \,, \tag{4.15}$$

which we prove here. The right-hand side of (4.15) can be written as

$$\lim_{t_1\to\infty}\int_{t_2=0}^{t_1}\int_{t_3=0}^{t_2}\int_{t_4=0}^{t_2} g(t_1-t_4)f(t_2-t_3)\,dt_2\,dt_3\,dt_4$$

and with the change of variables $x = t_1 - t_4$, $y = t_1 - t_2$, $z = t_2 - t_3$, this becomes

$$\lim_{t_1 \to \infty} \int_{x=0}^{t_1} g(x) \left[\int_{y=0}^{x} \left(\int_{z=0}^{t_1 - x} f(z)\, dz \right) dy \right] dx .$$

If $\int_0^\infty |f(x)|\, dx$, $\int_0^\infty |g(x)|\, x\, dx$ exist, the limit $t_1 \to \infty$ can be taken inside the integral, by Lebesgue's dominated convergence theorem, and (4.15) follows.

<u>REMARK</u>. Notice that $G^{(4)}$ vanishes if all the X_i commute with each other. Indeed, in that case $A(t)$ commutes with $A(s)$ for all t, s the solution of the equation (1.1) is

$f^\lambda(t) = \exp\left[\lambda \int_0^t A(s)\, ds \right] f_0$, and $P_0 f^\lambda(t)$

can be computed as in the first example of §1, where only $G^{(2)}$ appears.

Equation (1.16), describing the evolution of a polarization vector in a random magnetic field, can be treated in this way. More generally, an equation of the form (1.16) has been considered in [4] as an equation on $L^2(\Omega, \mu; \mathbb{R}^{2j+1})$ where J_1, J_2, J_3 are skew-symmetric generators of the $(2j+1)$ - dimensional irreducible representation of the rotation group, satisfying

$[J_i, J_j] = -\varepsilon_{ijk} J_k$ and $J_1^2 + J_2^2 + J_3^2 = j(j+1)\mathbb{1}$.

Then the equation describes rotational Brownian motion of a spherically symmetric molecule. The formulas (4.12) - (4.14) can be specialized to this case by letting

$k_{ij}(t) = \delta_{ij} k_j(t)$ (or even $\delta_{ij} k(t)$, $k(t)$ independent of j),

$X_i = J_i$, $i = 1, 2, 3$,

and using the algebraic relations among the J's. The explicit result is found in [4]; for $j = 1$ and $k(t) = \exp(-\alpha|t|)$ the result is quoted at the end of §1 -

APPENDIX Proof of the estimate (4.8):

Using the explicit form of $A(t)$ and the Gaussian property, we obtain

$$P_0 A(t_0) A(t_1) \ldots A(t_{2n}) A(t_{2n+1}) P_0$$

$$= \sum_{j_0,\ldots,j_{2n+1}=1}^{r} X_{j_0} X_{j_1} \ldots X_{j_{2n}} X_{j_{2n+1}} \langle \phi_{j_0}(t_0) \phi_{j_1}(t_1) \ldots \phi_{j_{2n}}(t_{2n}) \phi_{j_{2n+1}}(t_{2n+1}) \rangle$$

$$= \sum_{j_0,\ldots,j_{2n+1}=1}^{r} X_{j_0} \ldots X_{j_{2n+1}} \sum_{p \in \mathcal{P}_n''} \prod_{q=0}^{n} \langle \phi_{j_{p(2q)}}(t_{p(2q)}) \phi_{j_{p(2q+1)}}(t_{p(2q+1)}) \rangle$$

where $\mathcal{P}_n''$ is the set of those permutations of $\{0,1,\ldots,2n,2n+1\}$ such that $p(2q) < p(2q+1)$, $p(2q) < p(2q+2)$ for all $q = 0,1,\ldots,n-1$.

Inserting the projections $P_1 = 1 - P_0$ gives

$$A_{01}(t_0) A_{11}(t_1) \ldots A_{11}(t_{2n}) A_{10}(t_{2n+1})$$

$$= \sum_{j_0,\ldots,j_{2n+1}=1} X_{j_0} \ldots X_{j_{2n+1}} \sum_{p \in \mathcal{P}_n'} \prod_{q=0}^{n} \langle \phi_{j_{p(2q)}}(t_{p(2q)}) \phi_{j_{p(2q+1)}}(t_{p(2q+1)}) \rangle$$

where $\mathcal{P}_n'$ is the subset of $\mathcal{P}_n''$ consisting of those permutations in $\mathcal{P}_n''$ satisfying the additional condition that, for each $m = 1,\ldots,2n$, there is at least a $\bar{q} = \bar{q}(m)$ in $\{0,\ldots,n-1\}$ such that

$$p(2\bar{q}) \leq m, \; p(2\bar{q}+1) > m.$$

Let $I(t_0,\ldots,t_{2n+1}) = \| A_{01}(t_0) A_{11}(t_1) \cdots A_{11}(t_{2n}) A_{10}(t_{2n+1}) \|$.

We find

$$I(t_0,\ldots,t_{2n+1}) \leq (\kappa r^2 \|X\|^2)^{n+1} \sum_{p \in \mathcal{P}_n'} \exp\left[-\alpha \sum_{q=0}^{2n+1} (-1)^q t_{p(q)}\right]$$

$$= (\kappa r^2 \|X\|^2)^{n+1} \sum_{p \in \mathcal{P}_n'} \exp\left[-\alpha \sum_{k=0}^{2n+1} (-1)^{p^{-1}(k)} t_k\right], \qquad \text{(A.1)}$$

where we have used the bound (4.3) on the two-point functions. To each permutation p in $\mathcal{P}_n'$ there corresponds an arrangement of

signs $(-(-1)^{p^{-1}(0)}, \ldots, -(-1)^{p^{-1}(2n+1)})$; different permutations may give rise to the same arrangement. Let $f_p(k) = -(-1)^{p^{-1}(k)}$.

The possible arrangements of signs correspond to functions

$$f : \{0,\ldots,2n+1\} \to \{-1,+1\} \quad \text{such that} \quad \sum_{k=0}^{2n+1} f(k) = 0, \; \sum_{k=0}^{m} f(k) < 0$$

for all $m = 0,1,\ldots,2n$.

Denote by $F(n)$ the set of such functions, and for each f in $F(n)$ let $N(f)$ be the number of those permutations p in $\mathcal{P}_n'$ such that $f_p = f$, then we have

$$I(t_0,\ldots,t_{2n+1}) \leq (\kappa r^2 \|X\|^2)^{2n+1} \sum_{f \in F(n)} N(f) \exp\left[\alpha \sum_{k=0}^{2n+1} f(k) t_k\right].$$

Let $\mathcal{P}_n$ be the set of those permutations of $\{1,\ldots,2n\}$ such that $p(2q-1) < p(2q)$, $p(2q-1) < p(2q+1)$ for all q.

For each p in $\mathcal{P}_n$ there is exactly one f in $F(n)$ such that $(-1)^{p^{-1}(k)} = f(k)$ for all $k = 1,\ldots,2n$; for each f in $F(n)$, let $M(f)$ be the number of those permutations in $\mathcal{P}_n$ related to f in this manner.

Let also $C(n) = \max\{N(f)/M(f) : f \in F(n)\}$. Then we have

$$I(t_0,\ldots,t_{2n+1}) \leq C(n) (\kappa r^2 \|X\|^2)^{n+1} \sum_{f \in F(n)} M(f) \exp\left[\alpha \sum_{k=0}^{2n+1} f(k) t_k\right]$$

$$= C(n) (\kappa r^2 \|X\|^2)^{n+1} \exp[-\alpha(t_0 - t_{2n+1})] \sum_{p \in \mathcal{P}_n} \prod_{q=1}^{n} \exp[-\alpha(t_{p(2q-1)} - t_{p(2q)})]. \qquad \text{(A.2)}$$

Now we evaluate $C(n)$. For each f in $F(n)$, let $\{k_i(f) : i = 1,\ldots,n+1\}$ be the integers such that $f(k_i(f)) = +1$, arranged in increasing order. Notice that $k_1(f) \geq 2$ and $k_n(f) = 2n$, $k_{n+1}(f) = 2n+1$ for all f.

We compute $M(f)$ as follows: $k_1(f)$ may be associated by $p \in \mathcal{P}_n$ with $1, 2, \ldots, k_1(f) - 1$, giving $k_1(f) - 1$

possible pairings. Let $l_1(f,p) < k_1(f)$ be the integer associated with $k_1(f)$ by p. Then $k_2(f)$ may be associated with $1, 2, \ldots, \widehat{l_1(f,p)}, \ldots, \hat{k}_1(f), \ldots, k_2(f)-1$, giving $k_2(f)-3$ possible pairings.

Iterating the operation gives

$$M(f) = \prod_{j=1}^{n} (k_j(f) - 2j + 1).$$

With a similar procedure, we find an upper bound for $N(f)$:

$$N(f) \leqslant \prod_{j=1}^{n+1} (k_j(f) - 2j + 2),$$

since $p \in \mathcal{P}_n'$ may associate $k_i(f)$ with 0 as well; notice that the $(n+1)$-th factor in the product is 1 and may be omitted.

Let $m_j(f) = k_j(f) - 2j + 1$; then $m_j(f) \geqslant 1$ for all j and for all f, and there is a function $\bar{f}_n$ such that $m_j(\bar{f}_n) = 1$ for all j (it is given by $\bar{f}_n(k) = (-1)^k$ for all $k = 1, \ldots, 2n$). •

Then we find

$$C(n) \leqslant \max\left\{ \prod_{j=1}^{n} [1 + m_j(f)^{-1}] : f \in F(n) \right\} = 2^n. \tag{A.3}$$

(Actually, it is possible to show by induction on n that $N(\bar{f}_n) = 2^n$, $M(\bar{f}_n) = 1$, so that $C(n)$ equals 2^n).

Let now $n \geqslant 2$, $t_1 \geqslant \cdots \geqslant t_{2n}$. For each p in $\mathcal{P}_n$ let k_p, respectively l_p, be the element of $\{1, \ldots, 2n\}$ which is paired with $2n$, respectively $2n-1$; by the association of $p(2q-1)$ with $p(2q)$ for all q; and let $\bar{p}$ be the permutation in $\mathcal{P}_n$ which is obtained from p by pairing k_p with l_p, $2n-1$ with $2n$, and leaving all other pairings unaltered. Then $\bar{p}$ maps $\{1, \ldots, 2n-2\}$

into itself and $\bar{p} \restriction \{1, \ldots, 2n-2\}$ is an element of $\mathcal{P}_{n-1}$. For each $k = 1, \ldots, 2n-1$, let $\mathcal{P}_n^{(k)}$ be the set of those permutations in $\mathcal{P}_n$ such that $k_p = k$; the map $p \mapsto \bar{p} \restriction \{1, \ldots, 2n-2\}$ is a one-to-one map of $\mathcal{P}_n^{(k)}$ onto $\mathcal{P}_{n-1}$ for all k. If $k_p = 2n-1$, then $l_p = 2n$ and $\bar{p} = p$; for all p in $\mathcal{P}_n^{(k)}$, $k = 1, \ldots, 2n-2$, we have $l_p \neq 2n-1, 2n$ and

$$\prod_{q=1}^{n} \exp[-\alpha(t_{p(2q-1)} - t_{p(2q)})] = \prod_{q=1}^{n} \exp[-\alpha(t_{\bar{p}(2q-1)} - t_{\bar{p}(2q)}) - \alpha(t_{k_p \wedge l_p} - t_{2n-1})]$$

Since $\max\{k_p, l_p\} \equiv k_p \wedge l_p < 2n-1$, $t_1 \geqslant \cdots \geqslant t_{2n}$, $\alpha > 0$, it follows that

$$\prod_{q=1}^{n} \exp[-\alpha(t_{p(2q-1)} - t_{p(2q)})] \leqslant \prod_{q=1}^{n} \exp[-\alpha(t_{\bar{p}(2q-1)} - t_{\bar{p}(2q)})]$$

for all p in $\mathcal{P}_n$. By the one-to-one correspondence of $\mathcal{P}_n^{(k)}$ with $\mathcal{P}_{n-1}$ for each $k = 1, \ldots, 2n-1$, it follows that

$$\sum_{p \in \mathcal{P}_n} \prod_{q=1}^{n} \exp[-\alpha(t_{p(2q-1)} - t_{p(2q)})]$$
$$\leqslant (2n-1) \exp[-\alpha(t_{2n-1} - t_{2n})] \sum_{p \in \mathcal{P}_{n-1}} \prod_{q=1}^{n} \exp[-\alpha(t_{p(2q-1)} - t_{p(2q)})]. \tag{A.4}$$

For each $\varepsilon > 0$ there is a constant $K_0(\varepsilon)$ such that $(2n-1) \leqslant 2K_0(\varepsilon)(1+\varepsilon)^n$. Then, combining the estimates (A.2), (A.3), (A.4), we find the desired result (4.8).

REFERENCES

1. Davies, E.B.: Markovian master equations. Commun. math. Phys. 39, 91-110 (1974) - II. Math. Ann. 219 147-158 (1976) - III. Ann. Inst. H. Poincaré B 11, 265-273 (1975).

2 Papanicolaou, G.C. and Varadhan, S.R.S.: A limit theorem with strong mixing in Banach space and two applications to stochastic differential equations. Commun. Pure and Appl. Math. 26, 497-524 (1973).

3. Spohn, H.: Kinetic equations from Hamiltonian dynamics: Markovian limits. Rev. Mod. Phys. 52, 569-615 (1980).

4. Ford, G.W., Lewis, J.T. and McConnell, J.R.: Graphical study of rotational Brownian motion. Proc. R. Ir. Acad. 76 A, 117-143 (1976); Rotational Brownian motion of an asymmetric top. Phys. Rev. A 19, 907-919 (1979).

5. Frigerio, A., Lewis, J.T., and Pulè, J.V.: The averaging method for asymptotic evolutions II: Quantum open systems, in preparation.

6. Lewis, J.T., McConnell, J.R., and Scaife, B.K.P.: Relaxation effects in rotational Brownian motion. Proc. R. Ir. Acad. 76 A, 43-69 (1976)

7. Arnold, V.I.: Mathematical methods of classical mechanics. New York, Heidelberg, Berlin: Springer 1978.

8. Bogoliubov, N.N., and Mitropolsky, Y.A.: Asymptotic methods in the theory of nonlinear oscillations. New York: Gordon and Breach, 1961.

9. Case, K.M.: A general perturbation method for quantum mechanical problems. Prog. Theor. Phys. Suppl. 37. 1-20 (1966).

10. Ford, G.W.: Notes on stochastic differential equations (unpublished).

11. Ford, G.W., and Lewis, J.T.: A Bethe-type calculation of the Lamb shift for the harmonic oscillator, in preparation.

12. Palmer, P.F.: D.Phil. Thesis, Oxford 1976.

13. Kato, T.: Perturbation theory for linear operators. Berlin, Heidelberg, New York: Springer 1966.

14. Pulè, J.V.: The Bloch equations. Commun. math. Phys. 38, 241-256 (1974).